Linear Algebra II

05/07/2013, Thursday, 09:00-12:00

1 (13+5=18 pts) **Gram-Schmidt process**

Consider the vector space $C[-1,1]$, i.e. the vector space of continuous functions defined on the interval $[-1,1]$, and the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) By applying the Gram-Schmidt process, find an orthonormal basis for the subspace spanned by $\{1, x, x^2\}$.

(b) Find the coordinates of the function $1 + x^2$ in the orthonormal basis obtained above.

Required Knowledge: **inner product, Gram-Schmidt process.**

Solution:

(1a):

We begin with computing the involved inner products:

$$\langle 1,1\rangle = \int_{-1}^{1} 1\cdot 1\,dx = x\,|_{-1}^{1} = 2$$

$$\langle 1,x\rangle = \int_{-1}^{1} 1\cdot x\,dx = \frac{x^2}{2}\,|_{-1}^{1} = 0$$

$$\langle 1,x^2\rangle = \int_{-1}^{1} 1\cdot x^2\,dx = \frac{x^3}{3}\,|_{-1}^{1} = \frac{2}{3}$$

$$\langle x,x\rangle = \int_{-1}^{1} x\cdot x\,dx = \frac{x^3}{3}\,|_{-1}^{1} = \frac{2}{3}$$

$$\langle x,x^2\rangle = \int_{-1}^{1} x\cdot x^2\,dx = \frac{x^4}{4}\,|_{-1}^{1} = 0$$

$$\langle x^2,x^2\rangle = \int_{-1}^{1} x^2\cdot x^2\,dx = \frac{x^5}{5}\,|_{-1}^{1} = \frac{2}{5}.$$

By applying the Gram-Schmidt process, we obtain:

$$u_1 = \frac{1}{\|1\|}$$

$$u_1 = \frac{1}{\sqrt{2}}$$

$$u_2 = \frac{x - p_1}{\|x - p_1\|} \qquad p_1 = \langle x, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} = 0$$

$$\begin{aligned} x - p_1 &= x \\ \|x - p_1\|^2 &= \langle x, x \rangle \\ &= \frac{2}{3} \\ \|x - p_1\| &= \frac{\sqrt{2}}{\sqrt{3}} \end{aligned}$$

$$u_2 = \frac{\sqrt{3}}{\sqrt{2}} x$$

$$u_3 = \frac{x^2 - p_2}{\|x^2 - p_2\|} \qquad p_2 = \langle x^2, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} + \langle x^2, \frac{\sqrt{3}}{\sqrt{2}} x \rangle \frac{\sqrt{3}}{\sqrt{2}} x = \frac{1}{3}$$

$$\begin{aligned} x^2 - p_2 &= x^2 - \frac{1}{3} \\ \|x^2 - p_2\|^2 &= \langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle \\ &= \langle x^2, x^2 \rangle - 2\langle x^2, \frac{1}{3} \rangle + \langle \frac{1}{3}, \frac{1}{3} \rangle \\ &= \frac{2}{5} - 2\frac{1}{3}\frac{2}{3} + \frac{2}{9} \\ &= \frac{8}{45} \\ \|x^2 - p_2\| &= \frac{2\sqrt{2}}{3\sqrt{5}} \end{aligned}$$

$$u_3 = \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3}).$$

(1b): We have

$$\begin{aligned} 1 + x^2 &= \langle 1 + x^2, u_1 \rangle u_1 + \langle 1 + x^2, u_2 \rangle u_2 + \langle 1 + x^2, u_3 \rangle u_3 \\ &= \langle 1 + x^2, \frac{1}{\sqrt{2}} \rangle u_1 + \langle 1 + x^2, \frac{\sqrt{3}}{\sqrt{2}} x \rangle u_2 + \langle 1 + x^2, \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3}) \rangle u_3 \\ &= (\frac{2}{\sqrt{2}} + \frac{2}{3\sqrt{2}}) u_1 + 0 + \frac{3\sqrt{5}}{2\sqrt{2}} \langle 1 + x^2, x^2 - \frac{1}{3} \rangle u_3 \\ &= \frac{4\sqrt{2}}{3} u_1 + \frac{3\sqrt{5}}{2\sqrt{2}} (\frac{2}{3} - \frac{2}{3} + \frac{2}{5} - \frac{2}{9}) u_3 \\ &= \frac{4\sqrt{2}}{3} u_1 + \frac{2\sqrt{2}}{3\sqrt{5}} u_3. \end{aligned}$$

2 (2+4+4+4=14 pts) **Eigenvalues**

Let $M \in \mathbb{R}^{4\times 4}$ with the characteristic polynomial $p_M(\lambda) = \lambda^4 - 1$.

(a) Is M nonsingular? Why?

(b) Is M symmetric? Why?

(c) Is M diagonalizable? Why?

(d) Show that $M^{-2} = M^2$.

Required Knowledge: **eigenvalues, eigenvectors, diagonalization, Cayley-Hamilton theorem.**

Solution:

(2a): A matrix M is nonsingular if and only if zero is not one of its eigenvalues, that is $p_M(0) \neq 0$. Note that $p_M(0) = -1$. Hence, the matrix M is nonsingular.

(2b): All eigenvalues of a symmetric matrix are real numbers. The eigenvalues can be found by solving the equation $p_M(\lambda) = 0$. This results in $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = -i$, and $\lambda_4 = i$. As such, the matrix M cannot be symmetric.

(2c): A sufficient condition for diagonalizability is to have distinct eigenvalues. Then, it follows from the previous subproblem that the matrix M is diagonalizable.

(2d): We know from the Cayley-Hamilton theorem that $p_M(M) = 0$. This means that $M^4 = I$. By multiplying both sides by M^{-2}, we obtain $M^2 = M^{-2}$.

3 $\big((2+8)+(3+4+5)=22 \text{ pts}\big)$ **Positive definiteness**

(a) Consider the function

$$f(x,y,z) = -\frac{1}{4}(x^{-4}+y^{-4}+z^{-4}) + yz - x - 2y - 2z.$$

  (i) Verify that $(1,1,1)$ is a stationary point.

  (ii) Determine whether this point is a local minimum, local maximum, or saddle point.

(b) Let A be a symmetric positive definite matrix and B be a symmetric nonsingular matrix. Show that

  (i) A is nonsingular.

  (ii) A^{-1} is positive definite.

  (iii) $B^2 - 2I + B^{-2}$ is positive semi-definite.

REQUIRED KNOWLEDGE: **positive/definite matrices, leading principal minor test for positive definiteness.**

SOLUTION:

(3a-i): Let's find the partial derivatives with respect to the variables:

$$\frac{\partial f}{\partial x} = x^{-5} - 1 \qquad \frac{\partial f}{\partial y} = y^{-5} + z - 2 \qquad \frac{\partial f}{\partial z} = z^{-5} + y - 2.$$

Note that

$$\frac{\partial f}{\partial x}\Big|_{(1,1,1)} = \frac{\partial f}{\partial y}\Big|_{(1,1,1)} = \frac{\partial f}{\partial z}\Big|_{(1,1,1)} = 0.$$

Therefore, $(1,1,1)$ is a stationary point.

(3a-ii): To decide the nature of this stationary point, we look at the Hessian matrix:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}_{(1,1,1)} = \begin{bmatrix} -5x^{-6} & 0 & 0 \\ 0 & -5y^{-6} & 1 \\ 0 & 1 & -5z^{-6} \end{bmatrix}_{(1,1,1)} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & -5 \end{bmatrix}.$$

Since its diagonal elements are not positive, H cannot be positive definite. To check whether it is negative definite, we can check positive definiteness of $-H$. Note that

$$-H = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix}.$$

This matrix is positive definite as the principal minors are, respectively, 5, 25, and 120. Therefore, the Hessian is negative definite and the stationary point $(1,1,1)$ corresponds to a local maximum.

(3b-i): Since the matrix A is symmetric positive definite, all its eigenvalues must be positive. Therefore, zero is not an eigenvalue of A. Consequently, A is nonsingular.

(3b-ii): Let x be a nonzero vector. Note that

$$x^T A^{-1} x = y^T A y$$

where $x = Ay$. Since A is nonsingular, y is nonzero. Then, it follows from positive definiteness of A that

$$x^T A^{-1} x > 0$$

for any nonzero vector x. Consequently, A^{-1} is positive definite.

(3b-iii): Note that $B^2 - 2I + B^{-2} = (B - B^{-1})(B - B^{-1})$. Since B is symmetric, we have

$$B^2 - 2I + B^{-2} = (B - B^{-1})^T (B - B^{-1})$$

and hence

$$x^T (B^2 - 2I + B^{-2}) x = x^T (B - B^{-1})^T (B - B^{-1}) x = \|(B - B^{-1}) x\|^2 \geqslant 0.$$

This means that $B^2 - 2I + B^{-2}$ is positive semi-definite.

4 (10+(3+5)=18 pts) **Singular value decomposition**

(a) Find the singular value decomposition of the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Consider the decomposition

$$\begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

(i) Is this a singular value decomposition? Why?

(ii) Find the closest (with respect to the Frobenius norm) matrices of rank 1 and 2.

Required Knowledge: **singular value decomposition, lower rank approximations.**

Solution:

(4a):

Note that

$$A^T A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then, the characteristic polynomial of $A^T A$ can be found as:

$$p(\lambda) = \det(A^T A - \lambda I) = -(\lambda - 4)\big((\lambda - 5)^2 - 16\big).$$

This means that the eigenvalues of $A^T A$ are

$$\lambda_1 = 9 \qquad \lambda_2 = 4 \qquad \lambda_3 = 1.$$

Hence, the singular values of A are given by

$$\sigma_1 = 3 \qquad \sigma_2 = 2 \qquad \sigma_3 = 1.$$

To diagonalise $A^T A$, we need to compute its eigenvectors. For $\lambda_1 = 9$, we should solve:

$$(A^T A - 9I)v_1 = \begin{bmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -5 \end{bmatrix} v_1 = 0.$$

This leads to

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Similarly, by solving

$$(A^TA - 4I)v_2 = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2 = 0,$$

we obtain

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as the normalized eigenvector for $\lambda_2 = 4$. Finally, we solve

$$(A^TA - I)v_3 = \begin{bmatrix} 4 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} v_3 = 0$$

and obtain

$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Therefore, A^TA can be diagonalized by the orthogonal matrix

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix},$$

that is

$$\begin{bmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the number of non-zero singular values is 3, we have $\mathrm{rank}(A) = 3$. Then, we get

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$u_3 = \frac{1}{\sigma_3} A v_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

To obtain the last column of U, we need to find an orthonormal basis for the null space of A^T. Note that

$$0 = A^T x = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Therefore, we get

$$\mathcal{N}(A^T) = \mathrm{span}(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}).$$

This leads to

$$u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Consequently, a singular decomposition for A can be given as

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}.$$

(4b-i): Note that the two matrices

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

are orthogonal. Also, note that the matrix

$$\begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is of the required form for the singular value decomposition. As such, we can conclude that the given decomposition is a singular value decomposition.

(4b-ii): The best rank 1 approximation can be found as:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 4 & 4 & 2 \\ 4 & 4 & 2 \end{bmatrix}.$$

The rank 2 approximation is the matrix itself as it is of rank 2.

(a) Consider the matrix

$$\begin{bmatrix} a & b \\ 1 & a \end{bmatrix}$$

where a and b are real numbers. For which values of (a,b) is this matrix diagonalizable?

(b) Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that $\det(\lambda I - A) = (\lambda - 1)^3$. Put it into the Jordan canonical form.

Required Knowledge: **diagonalization, Jordan canonical form.**

Solution:

(5a): The characteristic polynomial for the matrix

$$\begin{bmatrix} a & b \\ 1 & a \end{bmatrix}$$

can be found as

$$p(\lambda) = (\lambda - a)^2 - b.$$

Observe that the eigenvalues are distinct if and only if $b \neq 0$. In this case, the matrix is diagonalizable. If $b = 0$, we have the matrix

$$\begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = \lambda_2 = a$. To find the eigenvector, one solves the equation:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x = 0.$$

Clearly, there is only one linearly independent eigenvector. Hence, this matrix is not diagonalizable.

Therefore, we arrive at the conclusion that the given matrix is diagonalizable if and only if $b \neq 0$.

(5b): Note that

$$\begin{aligned}
\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) &= -\lambda(1-\lambda)(2-\lambda) + (1-\lambda) \\
&= (1-\lambda)\Big(-\lambda(2-\lambda) + 1\Big) \\
&= (1-\lambda)(1 - 2\lambda + \lambda^2) \\
&= (1-\lambda)^3.
\end{aligned}$$

This proves the first part.

Note that the eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 1$. For the Jordan canonical form, we begin by finding out the eigenvectors:

$$0 = (A - I)v = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v.$$

This leads to, for instance, the two linearly independent eigenvectors:

$$x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Also note that $(A - I)^2 = 0$. This means that there will be two Jordan blocks. Finally, we need to check the feasibility of

$$(A - I)x' = x \qquad \text{and} \qquad (A - I)y' = y.$$

Note that

$$(A - I)x' = \begin{bmatrix} x_1' + x_2' \\ -x_1' - x_2' \\ 0 \end{bmatrix}.$$

Then, $(A - I)x' = x$ is not solvable. Also note that

$$(A - I)y' = \begin{bmatrix} y_1' + y_2' \\ -y_1' - y_2' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

yields, for instance,

$$y' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Consequently, we arrive the following Jordan canonical form:

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$